# TOPOLOGICAL PROPERTIES OF SPACES OF IDEALS OF THE MINIMAL TENSOR PRODUCT

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ABSTRACT. One shows that for two $C^*$-algebras $A_1$ and $A_2$ any continuous function on $\mathrm{Prim}(A_1)\times\mathrm{Prim}(A_2)$ can be continuously extended to $\mathrm{Prim}(A_1\otimes_{\min} A_2)$ provided it takes its values in a $T_1$ topological space. This generalizes [5, Corollary 3.4]. A new proof is given for a result of Archbold [2] about the space of minimal primal ideals of $A_1 \otimes_{\min} A_2$. To obtain these two results one makes use of the topological properties of the space of prime ideals of the tensor product.

## 1. Introduction and preliminaries

The prime ideal space of $A_1 \otimes A_2$, the minimal tensor product of two $C^*$-algebras $A_1$ and $A_2$, has some interesting topological properties in relation with the prime ideal spaces of the factors: there is a homeomorphism of $\mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2)$ onto a dense subset of $\mathrm{Prime}(A_1 \otimes A_2)$ and a continuous map of the latter space onto the first which, with the obvious identification, is a retract onto $\mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2)$. It turns out that these maps can be useful in getting information on the structure of $A_1 \otimes A_2$. Usually one employs the primitive ideal space to this end but since we do not know if a retraction as above exists in the case of $\mathrm{Prim}(A_1 \otimes A_2)$, the primitive ideal space of $A_1 \otimes A_2$, we have to use the prime ideal space instead.

By identifying the commutant of $A_1 \otimes \mathbf{1}$ in the multiplier algebra of $A_1 \otimes A_2$ Brown showed in [5, Corollary 3.4] that any bounded complex-valued continuous function on $\mathrm{Prim}(A_1) \times \mathrm{Prim}(A_2)$ has a continuous extension to $\mathrm{Prim}(A_1 \otimes A_2)$. The above mentioned retraction together with a device created by Kirchberg in [8] which completes a topological space with all its closed prime subsets allow us to find such an extension for every continuous function whose range is a $T_1$ topological space.

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Kaniuth proved in [7] that if $A_1 \otimes A_2$ has the property $(F)$ of Tomiyama then the minimal primal space (see below for the definition) of $A_1 \otimes A_2$ is canonically homeomorphic to Min-Primal$(A_1)\times$ Min-Primal$(A_2)$. Following that, Archbold proved in [2] that the same conclusion is valid in a more general situation than the presence of the property $(F)$. We give here a proof of this result of Archbold by using topological methods.

For a topological space $X$ we denote by $\mathcal{F}(X)$ the collection of all its closed subsets. We endow $\mathcal{F}(X)$ with the topology generated by all the families $\{F \in \mathcal{F}(X) \mid F \cap U \neq \emptyset\}$ where $U$ is an open subset of $X$. If $X$ is a $T_0$ space then the map $x \to \overline{\{x\}}$ is a homeomorphism of $X$ into $\mathcal{F}(X)$. A subset $L$ of $X$ is a limit set if there exists a net in $X$ that converges to all the points of $L$; by [6, Lemme 9] this is the same as saying that each finite collection of open subsets that intersect $L$ has a non void intersection. By Zorn's lemma every limit set is contained in a maximal (closed) limit set. The family of all maximal limit sets of $X$ is denoted $\mathcal{ML}(X)$ and will be considered with its relative topology inherited from $\mathcal{F}(X)$. A non void closed subset $F$ of $X$ is called prime if it is not the union of two closed subsets each different from $F$. Obviously, for each $x \in X$, $\overline{\{x\}}$ is prime. A space is called point-complete if each closed prime subset of it is the closure of a singleton. Following [8] we shall denote by $X^c$ the family of all closed prime subsets of a $T_0$ topological space $X$ endowed with the relative topology as a subfamily of $\mathcal{F}(X)$ and we shall call it the point-complete envelope of $X$. It is indeed a point-complete $T_0$ space. The base space $X$ will be identified with a subset of $X^c$.

Given a $C^*$-algebra $A$, an ideal of $A$ will always be a closed two sided ideal. We denote by $\mathrm{Id}(A)$ and $\mathrm{Id}'(A) := \mathrm{Id}(A) \setminus \{A\}$. The topology of the space of primitive ideals, $\mathrm{Prim}(A)$, is the usual hull-kernel topology and that of $\mathrm{Id}(A)$ is that one acquires by pulling the topology of $\mathcal{F}(\mathrm{Prim}(A))$ when one associates to each closed subset of $\mathrm{Prim}(A)$ its kernel. The relative topology of $\mathrm{Prime}(A)$ is also the hull-kernel topology and from here on by the hull of the ideal $I$, denoted hull$I$, we shall always mean the hull of $I$ in $\mathrm{Prime}(A)$. Clearly $\mathrm{Prime}(A)$ is $\mathrm{Prim}(A)^c$. An ideal $I$ of $A$ is called primal, cf. [3, Definition 3.1], if for every finite family of $\mathrm{Id}(A)$ with at least two members and zero product, $I$ contains one ideal of the family. An ideal is primal if and only if its hull is a closed limit set, see [1, Proposition 3.2]. There

the hull is taken in the primitive ideal space but the same proof works for prime ideals as well. Any primal ideal contains a minimal primal ideal (Zorn's lemma) and there is a one to one correspondence between the family of all minimal primal ideals, Min-Primal$(A)$, and $\mathcal{ML}(\mathrm{Prime}(A))$.

Let now $A_1$ and $A_2$ be $C^*$-algebras. For $I_j$ an ideal of $A_j$ we denote by $q_{I_j}$ the quotient map of $A_j$ onto $A_j/I_j$. One defines the maps $\Phi, \Delta : \mathrm{Id}(A_1) \times \mathrm{Id}(A_2) \to \mathrm{Id}(A_1 \otimes A_2)$ by

$$\Phi(I_1, I_2) := ker(q_{I_1} \otimes q_{I_2}), \quad \Delta(I_1, I_2) := I_1 \otimes A_2 + A_1 \otimes I_2.$$

Then $\Phi$ is a homeomorphism of $\mathrm{Id}'(A_1)\times\mathrm{Id}'(A_2)$ onto a dense subset of $\mathrm{Id}'(A_1\otimes A_2)$, see [9, Theorem 6]. Its restriction to $\mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2)$ maps it homeomorphically onto a dense subset of $\mathrm{Prime}(A_1 \otimes A_2)$, see [4, Lemma 2.13(v)] and [9, Corollary 8]. For $I$ an ideal of $A_1 \otimes A_2$ one defines

$$I_{A_1} := \{a_1 \in A_1 \mid a_1 \otimes A_2 \subset I\}, \quad I_{A_2} := \{a_2 \in A_2 \mid A_1 \otimes a_2 \subset I\}$$

and $\Psi(I) := (I_{A_1}, I_{A_2})$. Then $\Psi : \mathrm{Id}(A_1 \otimes A_2) \to \mathrm{Id}(A_1) \times \mathrm{Id}(A_2)$ is continuous and $\Psi \circ \Phi$ restricted to $\mathrm{Id}'(A_1) \times \mathrm{Id}'(A_2)$ is the identity map, see [9, proof of Theorem 6]. By this and [4, Lemma 2.13] $\Psi$ maps $\mathrm{Prime}(A_1 \otimes A_2)$ onto $\mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2)$.

## 2. Extensions of continuous functions

We begin with a simple lemma on extensions of continuous functions from a topological space to its point-complete envelope.

**Lemma 1.** *Let $X$ be a $T_0$ topological space and $f$ a continuous function from $X$ into a $T_1$ space $Y$. Then $f$ has a (unique) continuous extension from $X^c$ to $Y$.*

*Proof.* The function $f$ is constant on any prime closed subset of $X$. Indeed, if $S$ is such a subset and $f$ assumes two different values $y_1 \neq y_2$ on $S$ then we choose open neighbourhoods $V_1$, $V_2$ of $y_1$, $y_2$ respectively such that $y_1 \notin V_2$ and $y_2 \notin V_1$. Set now $S_1 := S \cap f^{-1}(Y \setminus V_1)$ and $S_2 := S \cap f^{-1}(Y \setminus V_2)$ and $\{S_1, S_2\}$ is a non-trivial decomposition of $S$.

We define now $\tilde{f} : X^c \to Y$ by $\tilde{f}(S) := f(x)$ for $x \in S$. Then $\tilde{f}$ is well defined and it is an extension of $f$. For $U$ an open subset of $Y$ we have

$$\{S \in X^c \mid \tilde{f}(S) \in U\} = \{S \in X^c \mid S \cap f^{-1}(U) \neq \emptyset\}$$

and the continuity of $\tilde{f}$ is established. □

We come now to the generalization of [5, Corollary 3.4]. There the the functions were considered on the spectra of the algebras; we prefer to work with the spaces of primitive ideals but, of course, there is no difficulty in obtaining a version of the following result in terms of spectra.

**Theorem 2.** *Let $A_1$ and $A_2$ be $C^*$-algebras and $\Phi : \mathrm{Id}'(A_1)\times\mathrm{Id}'(A_2) \to \mathrm{Id}'(A_1\otimes A_2)$ be the canonical homeomorphism. Then for every $T_1$ topological space $Y$ and any continuous function $f : (\mathrm{Prim}(A_1) \times \mathrm{Prim}(A_2)) \to Y$, the function $f \circ \Phi^{-1} : \Phi(\mathrm{Prim}(A_1)\times\mathrm{Prim}(A_2)) \to Y$ has a (unique) continuous extension from $\mathrm{Prim}(A_1\otimes A_2)$ to $Y$.*

*Proof.* Lemma 1 yields a continuous extension $\tilde{f} : (\mathrm{Prim}(A_1) \times \mathrm{Prim}(A_2))^c \to Y$ of $f$. By [8, Proposition 7.9] there is a homeomorphism $\nu$ from $\mathrm{Prim}(A_1)^c \times \mathrm{Prim}(A_2)^c = \mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2)$ onto $(\mathrm{Prim}(A_1) \times \mathrm{Prim}(A_2))^c$ which is the identity on the copies of $\mathrm{Prim}(A_1)\times\mathrm{Prim}(A_2)$ contained in these two spaces. Now, with $\Psi : \mathrm{Id}'(A_1 \otimes A_2) \to \mathrm{Id}'(A_1) \times \mathrm{Id}'(A_2)$ defined as in Section 1, the function $\tilde{f}\circ\nu\circ\Psi : \mathrm{Prime}(A_1\otimes A_2) \to Y$ is continuous. The extension $\hat{f}$ which we need is the restriction of $\tilde{f}\circ\nu\circ\Psi$ to $\mathrm{Prim}(A_1\otimes A_2)$. Indeed, if $(P_1, P_2) \in \mathrm{Prim}(A_1)\times\mathrm{Prim}(A_2)$ then $\hat{f}(\Phi(P_1, P_2)) = f(\nu(P_1, P_2)) = f(P_1, P_2)$ since $\Psi(\Phi(P_1, P_2)) = (P_1, P_2)$. □

## 3. Minimal primal ideals

In this section we present our proof for Archbold's result [2] on minimal primal ideals for tensor products. The first step is a topological lemma.

**Lemma 3.** *Let $X_1$, $X_2$, and $Y$ be topological spaces and $\phi$ a homeomorphism of $X_1 \times X_2$ onto a dense subset $Z$ of $Y$. Suppose there is a continuous map $\psi : Y \to X_1 \times X_2$ such that $\psi \circ \phi$ is the identity map of $X_1 \times X_2$ and for each $(M_1, M_2) \in \mathcal{ML}(X_1)\times\mathcal{ML}(X_2)$, $\psi^{-1}(M_1\times M_2)$ is the closure of $\phi(M_1\times M_2)$. Then $(M_1, M_2) \to \psi^{-1}(M_1 \times M_2)$ is a homeomorphism, $\Theta$ say, of $\mathcal{ML}(X_1) \times \mathcal{ML}(X_2)$ onto $\mathcal{ML}(Y)$.*

*Proof.* Obviously $\Theta$ is a one to one map.

Let $(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2)$. It is easily seen that $M_1 \times M_2$ is a closed limit set of $X_1 \times X_2$. Thus there exists a net in $Z$ that converges to all the points

of $M := \overline{\phi(M_1 \times M_2)} = \psi^{-1}(M_1 \times M_2)$. Suppose now that $\{y\} \cup M$ is a limit set of $Y$. $Z$ is dense in $Y$ hence there exists a net $\{s_\alpha\}$ in $X_1 \times X_2$ such that $\{\phi(s_\alpha)\}$ converges to all the points of $\{y\} \cup M$. Then $\{s_\alpha\}$ converges to all the points of $\psi(y) \cup \psi(M) = \psi(y) \cup (M_1 \times M_2)$. By using the canonical projections of $X_1 \times X_2$ onto the factors we infer from the maximality of the limit sets $M_1$ and $M_2$ that $\psi(y) \in M_1 \times M_2$ hence $y \in \psi^{-1}(M_1 \times M_2)$. We have shown that the map $\Theta$ takes its values in $\mathcal{ML}(Y)$.

Let now $L$ be a limit set in $Y$. As above, there is a net in $Z$ that converges to all the points of $L$ hence $\psi(L)$ is a limit set in $X_1 \times X_2$. Another use of the canonical projections of the cartesian product shows that there exist maximal limit sets $M_1$, $M_2$ in $X_1$, $X_2$, respectively, such that $\psi(L) \subset M_1 \times M_2$. Thus $L \subset \psi^{-1}(M_1 \times M_2)$. We have shown that each maximal limit set of $Y$ is in the image of $\Theta$.

If $U$ is an open subset of $Y$ then

$$\begin{aligned}
&\{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid \Theta(M_1, M_2) \cap U \neq \emptyset\} \\
&\quad = \{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid \overline{\phi(M_1 \times M_2)} \cap U \neq \emptyset\} \\
&\quad = \{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid \phi(M_1 \times M_2) \cap (U \cap Z) \neq \emptyset\} \\
&\quad = \{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid (M_1 \times M_2) \cap \phi^{-1}(U \cap Z) \neq \emptyset\}.
\end{aligned}$$

There exist open sets $\{V_\alpha^k\}$, $k = 1, 2$, such that $\phi^{-1}(U \cap Z) = \cup_\alpha (V_\alpha^1 \times V_\alpha^2)$. Thus

$$\begin{aligned}
&\{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid \Theta(M_1, M_2) \cap U \neq \emptyset\} \\
&\quad = \{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid (M_1 \times M_2) \cap (\cup_\alpha (V_\alpha^1 \times V_\alpha^2)) \neq \emptyset\} \\
&\quad = \{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid \cup_\alpha [(M_1 \times M_2) \cap (V_\alpha^1 \times V_\alpha^2)] \neq \emptyset\} \\
&\quad = \cup_\alpha [\{M_1 \in \mathcal{ML}(X_1) \mid M_1 \cap V_\alpha^1 \neq \emptyset\} \times \{M_2 \in \mathcal{ML}(X_2) \mid M_2 \cap V_\alpha^2 \neq \emptyset\}]
\end{aligned}$$

and the latter is an open set in $\mathcal{ML}(X_1) \times \mathcal{ML}(X_2)$. We conclude that $\Theta$ is continuous.

Let now $V_k$ be open in $X_k$, $k = 1, 2$; there exists an open set $W$ of $Y$ such that $\phi(V_1 \times V_2) = Z \cap W$. We have

$$\begin{aligned}
&\Theta(\{(M_1, M_2) \in \mathcal{ML}(X_1) \times \mathcal{ML}(X_2) \mid M_1 \cap V_1 \neq \emptyset, \quad M_2 \cap V_2 \neq \emptyset\}) \\
&\quad = \{\Theta(M_1, M_2) \in \mathcal{ML}(X_1 \times X_2) \mid \phi(M_1 \times M_2) \cap \phi(V_1 \times V_2) \neq \emptyset\} \\
&\quad = \{\Theta(M_1, M_2) \in \mathcal{ML}(X_1 \times X_2) \mid \overline{\phi(M_1 \times M_2)} \cap W \neq \emptyset\}
\end{aligned}$$

and this is an open subset of $\mathcal{ML}(X_1 \times X_2)$. Thus we obtained that $\Theta$ is open and this concludes the proof.

$\square$

**Lemma 4.** *Let $A_1$ and $A_2$ be $C^*$-algebras and $I_1$, $I_2$ ideals in $A_1$, $A_2$, respectively. Then* $\mathrm{hull}\Delta(I_1, I_2) = \Psi^{-1}(\mathrm{hull}I_1 \times \mathrm{hull}I_2)$ *and* $\mathrm{hull}\Phi(I_1, I_2) = \overline{\Phi(\mathrm{hull}I_1 \times \mathrm{hull}I_2)}$.

*Proof.* Suppose $P \in \mathrm{hull}\Delta(I_1, I_2)$; then $\Psi(P) = (P_{A_1}, P_{A_2}) \in \mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2)$ and $P_{A_1} \supseteq I_1$, $P_{A_2} \supseteq I_2$. Thus $P \in \Psi^{-1}(\mathrm{hull}I_1 \times \mathrm{hull}I_2)$. Conversely, if $P \in \mathrm{Prime}(A_1 \otimes A_2)$ and $\psi(P) \in \mathrm{hull}I_1 \times \mathrm{hull}I_2$ then $P \supseteq \Delta(P_{A_1}, P_{A_2}) \supseteq \Delta(I_1, I_2)$ and we got the reverse inclusion.

The second equality is [9, Corollary 3].

$\square$

The following result is an improvement obtained by Archbold of [7, Theorem 1.1].

**Theorem 5** (Theorem 4.1 of [2])**.** *Let $A_1$ and $A_2$ be $C^*$-algebras. If $\Phi(I_1, I_2) = \Delta(I_1, I_2)$ for all $(I_1, I_2) \in$ Min-Primal$(A_1) \times$ Min-Primal$(A_2)$ then $\Phi$ is a homeomorphism of* Min-Primal$(A_1) \times$ Min-Primal$(A_2)$ *onto* Min-Primal$(A_1 \otimes A_2)$.

*Proof.* We shall exploit the fact that for a $C^*$-algebra $A$, the map $\mathrm{hull}(I) \to I$ is a homeomorphism of $\mathcal{F}(\mathrm{Prime}(A))$ onto $\mathrm{Id}(A)$ that maps $\mathcal{ML}(\mathrm{Prime}(A))$ onto Min-Primal$(A)$. Thus the conclusion will be obtained once we show that $(M_1, M_2) \to \mathrm{hull}\Phi(\ker M_1, \ker M_2)$ is a homeomorphism of $\mathcal{ML}(\mathrm{Prime}(A_1)) \times \mathcal{ML}(\mathrm{Prime}(A_2))$ onto $\mathcal{ML}(\mathrm{Prime}(A_1 \otimes A_2))$.

By Lemma 4, the hypothesis on $A_1 \otimes A_2$ is $\overline{\Phi(M_1 \times M_2)} = \Psi^{-1}(M_1 \times M_2)$. Thus the maps $\Phi : \mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2) \to \mathrm{Prime}(A_1 \otimes A_2)$ and $\Psi : \mathrm{Prime}(A_1 \otimes A_2) \to \mathrm{Prime}(A_1) \times \mathrm{Prime}(A_2)$ satisfy the conditions of Lemma 3 which yields the desired homeomorphism.

$\square$

It is remarked in [2, p. 142] that there is no known example of $C^*$-algebras $A_1$, $A_2$ and minimal primal ideals $I_1$, $I_2$ of these algebras such that $\Phi(I_1, I_2) \neq \Delta(I_1, I_2)$. By contrast, one constructs easily an example of topological spaces $X_1$, $X_2$, and $Y$, a homeomorphism $\phi$ of $X_1 \times X_2$ onto a dense subset of $Y$, a continuous map

$\psi : Y \to X_1 \times X_2$ such that $\psi \circ \phi$ is the identity map of $X_1 \times X_2$ and maximal limit sets $M_1 \subset X_1$, $M_2 \subset X_2$ such that $\psi^{-1}(M_1 \times M_2) \neq \overline{\phi(M_1 \times M_2)}$.

*Example* 6. Let $X_1 = X_2 := [0, 1]$ with the usual topology and

$$Y := ([0, 1] \times [0, 1]) \cup \{y\}$$

where $y$ is a point not in the square. A base for the topology of $Y$ consists of the topology of the square together with the family of all the sets $(U \setminus \{v\}) \cup \{y\}$ where $v := (1, 1)$ and $U$ runs through all the open neighbourhoods of $v$. Let $\phi$ be the identity map of the square and $\psi$ the map that is the identity on the square and takes $y$ to $v$. For $M_1 = M_2 := \{1\}$ we have $\overline{M_1 \times M_2} = \{v\}$ but $\psi^{-1}(M_1 \times M_2) = \{v, y\}$.

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